Keywords: constant flux states; non-hermitian modes; purcell factor; steady-state laser theory

Constant flux (CF) states describe the steady state response of a photonic medium with an arbitrary, possibly frequency dependent index of refraction $n(x, \omega)$ to a harmonically oscillating source. CF states were initially introduced to describe steady state oscillations of complex lasers. Here we describe their application to various phenomena in photonics and quantum optics.
1. Introduction

The description of many phenomena in photonics relies on the idea of normal modes. The concept of normal modes is so powerful because it provides mathematically, numerically and conceptually the most economical way to describe light-matter interactions. For instance, both the semiclassical laser theory and quantum optics rely heavily on the expansion of the radiation field in normal modes that are suitable for the given given optical structure, be it a “cavity” or a continuous medium. Despite the ubiquity of normal mode description, when it comes to the description of open photonic structures, there is a plethora of alternative descriptions of what normal modes of the system should be. In open structures, the radiation can escape to infinity from the cavity, requiring either a non-hermitian description via (discrete) quasi-modes of a finite system\(^1\) or a hermitian description via the (continuous) modes of the universe of an infinite system\(^2\). Both descriptions have their advantages and disadvantages: While modes of the universe provide a mathematically consistent framework thanks to the complete understanding of the spectral problem of hermitian operators, they fail to focus on the details of the cavity; Quasi-modes on the other hand use the cavity structure as a point of departure but in general yield a problematic mathematical framework. We will focus here on quasi-modes, because they provide the most effective description and show that a consistent mathematical and computational framework can be reached with Constant Flux (CF) modes. Our aim in this chapter is to illustrate how seeming mathematical pathologies of quasi-modes are related to the time-independent (frequency-domain) language that is employed many times, and that everything falls into place when one considers the underlying problem that is fundamentally time-dependent, due to omnipresence of matter (sources) which emits or absorbs photons.

The structure of this chapter is as follows: In Section 2, we will analyze in depth the source-field problem in the most simple setting for a one dimensional (1D) dielectric cavity to illustrate the underlying physics. This system has the advantage that it contains a number of the generic features of open systems, while being mathematically transparent. In Section 3 we will discuss two applications of CF states, in semiclassical laser theory and quantum optics, respectively.
2. Source-field solutions for 1D cavity

Consider the field measured at $x$ at time $t$, $e(x,t)$, in response to a point source oscillating harmonically at the frequency $\Omega$ located at $x'$ inside a one-dimensional dielectric cavity

$$\left[ \partial_x^2 - n^2(x) \partial_t^2 \right] e(x,t) = f(x,t) \tag{1}$$

with $f(x,t) = \delta(x-x')e^{-i\Omega t}$. The “cavity” is defined by the discontinuity of the index of refraction (for simplicity we assume here a dispersionless medium) $n(x) = n_0$ for $0 < x < a$, and $n(x) = 1$ for $x > a$. Without a loss in generality, we assume that the cavity is terminated by a perfectly reflecting mirror at $x = 0$, thus $e(x=0,t) = 0$. The dielectric continuity conditions at $x=a$ are given by the continuity of $e(x,t)$ and $\partial_x e(x,t)$. Finally, the radiation generated by the source has to be flowing to spatial infinity, a condition expressed by outgoing boundary conditions for $x \to \infty$, given by $\partial_x e(x,t) = -\partial_t e(x,t)$. We set $c=1$, so that outside the cavity the field $e(x,t)$ has to oscillate in space at the wavevector $k = \Omega$ due to linear dispersion of vacuum. Fourier transforming Eq. (1) in time, we obtain

$$\left[ \frac{1}{n^2(x)} \partial_x^2 + \Omega^2 \right] \tilde{e}(x,\Omega) = \frac{1}{n^2(x)} \delta(x-x') \tag{2}$$

Consider the auxiliary eigenvalue problem of the Laplace operator $L = -\frac{1}{n^2(x)} \partial_x^2 - \frac{1}{n^2(x)} \partial_y^2 \varphi_m(x,\Omega) = \omega_m^2(\Omega) \varphi_m(x,\Omega) \tag{3}$

Note that we do not have to assign any physical meaning to these eigensolutions at this stage; they have to be calculated to solve the inhomogeneous problem (2), and hence by Fourier transform Eq. (1), which is the actual physical problem at hand.

The $\Omega$-dependence of the eigenfrequencies $\omega_m(\Omega)$ and eigenmodes $\varphi_m(x,\Omega)$ are due to the outgoing boundary conditions $\partial_x \varphi_m(x = \infty) = i\Omega \varphi_m(x = \infty)$, the eigenfunctions have to satisfy. The solutions are simply trigonometric functions

$$\varphi_m(x,\Omega) = \begin{cases} \sin(n_0 \omega_m(\Omega) x) & x < a \\ \sin(n_0 \omega_m(\Omega) a) e^{i\Omega(x-a)} & x > a \end{cases} \tag{4}$$

The eigenfrequencies $\omega_m(\Omega)$ are found through the characteristic equation

$$\tan(n_0 \omega_m a) = -i \frac{n_0 \omega_m}{\Omega} \tag{5}$$

and always have a non-zero imaginary part. In Fig. 1 we show the parametric dependence on $\Omega$ of the eigenfrequencies $\omega_m(\Omega)$.

In order to construct the solution to Eq. (1), we also need to solve the adjoint spectral problem. This is defined by the adjoint operator $L^* = -\frac{1}{(n^*)^2(\Omega)} \partial_y^2$ and an adjoint function space $\{ \varphi(x) \}$ with adjoint boundary conditions $\partial_x \varphi(x = \infty,\Omega) = -i\Omega \varphi(x = \infty,\Omega)$. The adjoint eigenfunctions are found to be

$$\bar{\varphi}_m(x,\Omega) = \begin{cases} \sin(n_0^* \omega_m(\Omega) x) & x < a \\ \sin(n_0^* \omega_m(\Omega) a) e^{-i\Omega(x-a)} & x > a \end{cases} \tag{6}$$
Fig. 1. Parametric dependence of the CF eigenvalues $\omega_m(\Omega)$. Solid lines show the variation of $\omega_m(\Omega)$ for $m = 1, 2, 3$ as $\Omega$ is varied in the interval $(0, 5)$. Note that all the eigenvalues are in the lower complex plane. We sample three of the points with the three dots (rectangle, circle, star) on each $\omega_m$ trace. The $\Omega$ values at which $\omega_m(\Omega)$ is sampled is highlighted on the real axis. For this plot, $a = 1$ and $n_0 = 2$.

The dual eigenfrequencies are found through the adjoint characteristic equation

$$\tan(n_0^* \tilde{\omega}_m a) = i \frac{n_0^* \tilde{\omega}_m}{\Omega}$$

By comparing to Eq. 6, one can easily see that $\tilde{\omega}_m = \omega_m^*$ and $\tilde{\varphi}_m = \varphi_m^*$. It can then be straightforwardly shown that

$$\langle \langle \varphi_m | \varphi_n \rangle \rangle = \int_0^a dx \tilde{\varphi}_m^*(x, \Omega) \varphi_n(x, \Omega) = \int_0^a dx \varphi_m(x, \Omega) \varphi_n(x, \Omega) = \delta_{mn} \eta_n(\Omega)$$

Thus the usual orthogonality relation only exists between the eigenfunctions of adjoint spaces.

The solution of (2) is proportional to the Green’s function $\tilde{e}(x, \Omega) = \frac{1}{\pi^2(x)} G(x, x'; \Omega)$, which can then be constructed by the solutions of the auxiliary eigenvalue problem and its adjoint:

$$G(x, x'; \Omega) = \sum_m \frac{\varphi_m(x, \Omega) \varphi_m^*(x', \Omega)}{\eta_m(\Omega)(\Omega^2 - \omega_m^2(\Omega))}$$

The normalization factor is given by

$$\eta_m(\Omega) = a \left( 2 - \frac{\sin(2n_0\omega_m(\Omega)a)}{4n_0\omega_m(\Omega)a} \right)$$

Finally, the solution $e(x, t)$ for an arbitrary time-dependent source term $f(x, t)$ can be calculated using the Green’s function found above:

$$e(x, t) = \int_0^a dx' \int_{-\infty}^{\infty} dY \int_{-\infty}^{\infty} dt' e^{-i\Omega'(t-t')} G(x, x'; \Omega') f(x', t')$$
To understand the physical meaning of the modes $\varphi_m$ consider a source distribution in Eq. (19) of the form $f(x,t) = \varphi_m(x,\Omega)e^{-i\Omega t}$ inside the cavity. The response to such a source distribution, using Eq. (13), is

$$e(x,t) = \frac{1}{n_0^2} \frac{\varphi_m(x,\Omega)}{\Omega^2 - \omega_m^2(\Omega)} e^{-i\Omega t}$$

(14)

showing that the states $\varphi_m(x,\Omega)$ correspond to special spatial distributions of harmonically oscillating sources at frequency $\Omega$ in the cavity ($0 < x < a$) that produce a field proportional to themselves. Another important feature of these modes is that the modes $\varphi_m(x,\Omega)$ defined here carry a constant flux to infinity ($j \sim \text{Im} \left[ \varphi_m(x) \partial_x \varphi_m^*(x) \right] = \Omega |\sin(n_0\omega_m(\Omega)a)|^2$ for $x > a$). These modes are therefore termed Constant Flux (CF) modes.

Note that the definition of a harmonic source implies that the sources are switched on in the dim past, at $t \to -\infty$, so that there is no transient response. Consider now a pulse-source of the form $f(x,t) = \delta(x-x')\delta(t)$, the resulting impulse-response can be obtained via Eq. (13):

$$e_\delta(x,t) = \int_{-\infty}^{\infty} d\Omega e^{-i\Omega t} G(x,x';\Omega)$$

(15)

Inspecting the Green's function (11) we find two sets of poles. First, the factor $[\Omega^2 - \omega_m^2(\Omega)]^{-2}$ has poles at $\Omega = +\omega_m(\Omega)$, which we denote by $\omega_m^{QB}$; these are the Quasi-bound (QB) modes of the system\(^4\). These poles lie in the lower half plane - note that no solution exists for $\Omega = -\omega_m(\Omega)$. The poles $\omega_m^{QB}$ can be calculated.
analytically and are given by\(^3\):
\[
\omega_m^{\alpha} \equiv \nu_m - i\kappa = \frac{1}{n_0 a} \left[ \pi (m + 1/2) - i \ln \left( \frac{n_0 + 1}{n_0 - 1} \right) \right]
\]
(16)

Second, the normalization factor \(\eta_m(\Omega)\) given in (12) has also poles in the lower half-plane, symmetric around the imaginary axis of \(\Omega\). Unfortunately, there is no simple analytical expression for these poles.

One widely encountered curiosity is that the QB modes \(\varphi_m^{\alpha}(x) = \varphi_m(\omega_m^{\alpha}, x)\) blow up at \(x = \infty\) as \(e^{ix}\); naively, one might expect that this divergence reflects the expectation that in the long-time limit all energy will have accumulated at \(x = \infty\) and that it doesn’t make sense to consider \(e(x,t) \propto e^{i(x-t)}\) (for \(x > a\)) beyond the ‘causality cone’ \(x > t\). However, a more careful analysis of the integral (15) shows that through a delicate destructive interference of the terms in Eq. (11), the divergence at \(x = \infty\) is removed and that the causality need not be imposed by hand.

Calculating the integral (15) by using appropriate contour integration and ignoring the poles resulting from zeros of \(\eta_m(\Omega)\), we obtain for \(x > a\)
\[
e_\delta(x,t) = -\theta(t - x + a - n_0(a - x')) \frac{2\pi i}{n_0 a} \sum_m \frac{\varphi_m^{\alpha}(x) (\varphi_m^{\alpha}(x')^*) e^{-i\omega_m^{\alpha} t}}{\omega_m^{\alpha}}
\]
\[
= \theta(t - x + a - n_0(a - x')) \frac{2\pi i}{n_0 a} \int_{-\infty}^{t} d\tau \sum_m \frac{\varphi_m^{\alpha}(x) (\varphi_m^{\alpha}(x')^*) e^{-i\omega_m^{\alpha} \tau}}{\omega_m^{\alpha}}
\]
\[
= \theta(t - x + a - n_0(a - x')) \frac{2\pi i}{n_0 a} \sum_k (-1)^k \left( \frac{n_0 - 1}{n_0 + 1} \right)^k \left[ \theta \left( t - (x - a) - n_0(a + x') - 2kn_0 a \right) - \theta \left( t - (x - a) - n_0(a + x') - 2kn_0 a \right) \right]
\]
(17)

Here, \(\theta(x)\) is the Heaviside step-function. The step-function outside the sum results from the choice of closing the integration contour either in the lower or upper-half of the complex \(\Omega\)-plane. Note that due to this prefactor, the entire function is non-zero only for \(t > L(x,x')\) and that \(L(x,x) = (x - a) + n_0(a - x')\) is the direct (without intermediate reflections) optical path from the source to the observation point. The factor \(r = (n_0 - 1)/(n_0 + 1)\) in the summation is immediately recognized as the Fresnel reflection coefficient, its powers \(r^k\) yielding the correct amplitude for a multiply reflected pulse. Figure 2 displays snapshots of the propagating pulse for different times. We find a pulse that initially spreads in both directions from the source at \(x'\). Upon the left front reaching the reflecting boundary at \(x = 0\) (assuming \(x' < a/2\)), the rectangular pulse remains at a width \(2x'\) and starts to propagate towards the interface at \(x = a\). Reaching it, the pulse partially reflects back into the cavity, and partially transmits a pulse of width \(2n_0 a x'\) to infinity. The reflected pulse propagates back towards the perfect mirror, at which the left front reflects.
and annihilates the pulse exactly at position $x'$. The field stays not zero at this point however, as the first derivative of the field amplitude will cause a new pulse to start expanding at $x'$ but with negative amplitude. A camera in the farfield would observe a train of pulses of width $2n_0 x'$ of decaying amplitudes and alternating sign (assuming a phase-sensitive detector). The envelope of the pulse-amplitudes decay as $e^{-\kappa t}$ starting with the first pulse that arrives at $x$ a time $\Delta t = x - a + n_0(a - x')$ after it is emitted at $x'$.

This at first sounds somewhat odd when we think of pulse propagation in uniform media with a frequency independent index of refraction, which we expect to be non-dispersive. The initial pulse $\delta(x - x')\delta(t)$ spreads and seems to stop spreading once the reflecting boundary is reached, reflecting back and forth after that. Such an expectation is based on our intuition about wave-propagation in three dimensional space. In reduced dimensions, because of the effective sources being extended (i.e. a point source in two dimensions is physically a line-source in 3D and so on), the fundamental solutions are different than those in 3D. While the 3D solution is an expanding spherical shell of electric field with its center at the source, the 2D solution is a front that has a trailing tail that decays as a power-law all the way back to the source, and the 1D solution is simply a uniform field with fronts on both ends expanding\(^4\). That a pulse started in a cavity described by Eq. (17) stops spreading once the reflecting wall is reached can be simply understood by utilizing method of images: image sources located periodically at $x = -x', x = -x' - 2a$, $x = -2x' - 2a$, $x = -2x' - 4a$, and so forth, do also emit spreading pulses that cancel the physical pulse at $x = 0$ to satisfy the BC there. The net effect is that the pulse width after reflection from $x = 0$ remains constant.

In conclusion, we see that QB modes are relevant only in describing transient response while CF states form a more general basis to describe any type of source; in fact we have derived (17) starting with the CF Green’s function (11). In the Appendix, we show that the impulse response calculated above fully agrees with what is obtained starting from the exact Green’s function. The latter can be calculated in a closed form in the 1D case.

Note that the CF modes in Eq. (4) all peak at the open interface. This can be easily understood in a semiclassical description: We want the ray amplitude and phase to be conserved through a roundtrip (the phase up to $2\pi$) and there is an amplitude loss at the open interface. This is only possible if the amplitude increases towards that interface. If we had two open interfaces, we would have a reflection symmetric distribution with higher amplitude at both interfaces.

Finally, we would like to highlight a physically appealing and useful interpretation of the CF modes. The quantization condition Eq. 6 can alternatively be thought in the following way. Define a frequency dependent effective index of refraction $n_m(\Omega) = n_0 \omega_m / \Omega$ which is complex. Then the quantization condition can be rewritten as

$$\tan(\Omega n_m(\Omega)) = -in_m(\Omega)$$

(18)
This is the equation for the quasi-bound modes of the system, with a frequency dependent index of refraction. Starting with the discrete quasi-bound modes enumerated by $m$ for a constant index of refraction $n_0$, imagine tuning the real and imaginary part of the index of refraction to $n_m(\Omega) = n_m^R(\Omega) + in_m^I(\Omega)$ so that we impart precisely the amplification and dispersion necessary for the mode $m$ to oscillate at the stationary frequency $\Omega$. The corresponding solutions are exactly the CF modes $\varphi_m(x, \Omega)$. In this manner, at each $\Omega$ we can construct a discrete set of modes which are continuously related to the original quasi-bound modes of the cavity with a constant index of refraction.

3. Applications of CF states

We will focus in this section on two applications of CF states in the semiclassical laser theory and quantum optics. For transparency of presentation we will focus in this section again on 1D cavities.

3.1. Steady-state semiclassical laser theory

The starting point of semiclassical laser theory is the set of Maxwell-Bloch (MB) equations, which describe the electric and magnetic fields using Maxwell’s equations, with a source term (polarization) that is generated by a set of uniformly distributed two-level emitters, described quantum mechanically. Using the rotating wave and stationary inversion approximations\(^3\) the MB equation can be reduced to the following source-field problem:

$$[\partial_x^2 - n^2(x)\partial_t^2] e(x,t) = f(x,t)$$  \hspace{1cm} (19)

where $e(x,t) = \sum_\mu \Psi_\mu(x)e^{-i\Omega_\mu t}$ and

$$f(x,t) = -\frac{D_0(x)}{1 + \sum_\nu |\Psi_\nu(x)|^2} \sum_{\mu=1}^N \left( \frac{\Omega_\mu}{\omega_a} \right)^2 \frac{\Psi_\mu(x)e^{-i\Omega_\mu t}}{-i(\Omega_\mu - \omega_a) + \gamma_\perp}$$

Here, $D_0(x)$ measures the pumped energy in terms of the inversion generated per unit volume in the absence of gain saturation ($e = 0$), $\Gamma_\nu = \Gamma(\Omega_\nu) = \gamma_\perp^2 / [(\Omega_\nu - \omega_a)^2 + \gamma_\perp^2]$ is the laser gain-curve, $\omega_a$ and $\gamma_\perp$ are the atomic resonance frequency and the associated homogeneous broadening, respectively. This problem is very similar to the problem given by Eq. (1) discussed in Section 2, with one notable difference: the source term is dependent on the field $e(x,t)$ itself via the unknown non-linear lasing modes $\Psi_\mu(x)$ and the (real) laser frequencies $\Omega_\mu$. This makes the problem Eqs. (19)-(20) non-linear requiring a self-consistent solution technique that we describe below to determine the solutions.

The formal solution is obtained by using the Green’s function methods of Sec-
tion 2 for harmonically oscillating sources, Eq. (1), via Eq. (13):

\[ \Psi_\mu(x) = i\gamma_\perp \frac{\Omega_\mu^2}{-i(\Omega_\mu - \omega_a) + \gamma_\perp \omega_a^2} \]

\[ \times \int_D dx' D_0(x') G(x, x'; \Omega_\mu) \Psi_\mu(x') \]

\[ n^2(x', \Omega_\mu) (1 + \sum_\nu \Gamma_\nu |\Psi_\nu(x')|^2)^2. \]

(20)

This set of non-linear integral equations for lasing modes \( \Psi_\mu(x) \) and the laser frequencies \( \Omega_\mu \) constitute the Steady-state Ab-initio Laser Theory (SALT) first described in Ref. 3. Note that the integral is limited to the domain \( D \), within which the gain medium is located; for simplicity we assume that the pump is uniform across the entire cavity \( 0 < x < a \) i.e. \( D_0(x) = D_0 \theta(a - x) \theta(x) \). In contrast to standard approaches, cavity losses enter this equation through the Green function and ultimately, as we have seen in Section 2, through the boundary conditions at infinity. Below we briefly outline the solution technique and a powerful numerical algorithm to determine the multi-mode lasing solutions at any pump-rate \( D_0 \).

The relations found in Section 2 make it possible to expand an arbitrary lasing solution in the form

\[ \Psi_\mu(x) = \sum_{m=1}^{\infty} a_{\mu}^m \varphi_\mu^m(x), \]

(21)

so that each \( \Psi_\mu(x) \) is defined by the vector of complex coefficients \( a_\mu \) in the space of CF states. In what follows we will use the shorthand \( \varphi_\mu^m(x) \equiv \varphi_\mu(x, \Omega_\mu) \) and \( \omega_\mu^m \equiv \omega_\mu(\Omega_\mu) \). Because only CF states with frequencies near the center of the gain curve contribute to the lasing state, it is possible to truncate the sum in Eq. (21) to a finite number (\( N \)) of components, making \( a_\mu \) a finite dimensional vector. By substitution of Eq. (21) into Eq. (20) and use of the biorthogonality relations (10) one finds:

\[ a_{\mu}^m = \frac{i D_0 \gamma_\perp}{(\gamma_\perp - i(\Omega_\mu - \omega_a)) (\Omega_\mu^2 - \omega_a^2)} \]

\[ \times \int_D dx' \frac{\varphi_\mu^*(x') \sum_\nu a_{\nu}^m \varphi_\nu(x')}{n^2(x', \Omega_\mu) (1 + \sum_\nu \Gamma_\nu |\Psi_\nu(x')|^2)^2}. \]

(22)

This is the form of Eq. (20) that is employed in SALT for finding the lasing modes and frequencies. As discussed above, it reduces the problem to finding the complex vector of coefficients \( a_\mu \) and the frequency \( \Omega_\mu \) for each lasing mode, which depends non-linearly on all the other lasing modes and itself through the infinite order non-linearity evident in the denominator of equation (22). Here we will focus on the near-threshold solutions of this set of equations. The technique for determining the multi-mode lasing solutions for any pump rate \( D_0 \) is described in Ref. 5.

For a pump rate below the first lasing threshold \( D_0 < D_{th} \), only the trivial solution exists: \( \Psi_\mu = 0, \forall \mu \). This is the non-lasing solution i.e. a standard lamp, which because of the neglect of the quantum fluctuations appears to have zero amplitude. Very close to but above the first threshold \( D_0 = D_{th} + \epsilon \), we may
neglect the term $\sum_{\nu} \Gamma_{\nu} |\Psi_{\nu}(x')|^2$ in the denominator of Eq. (22). The resulting set of equations can then be compactly written in the form of an eigenvalue equation

$$T^{(0)}(\Omega) a^\mu = \left(1/D_0\right) a^\mu .$$

(23)

where $\Lambda_m(\Omega) = i\gamma_{\perp}(\Omega^2/\omega_0^2)/[(\gamma_{\perp} - i(\Omega - \omega_0))(\Omega^2 - \omega_m^2(\Omega))]$ and $T^{(0)}(\Omega)$ is a matrix

$$T^{(0)}_{mn}(\Omega) = \Lambda_m(\Omega) \int dx' \frac{\bar{\varphi}_{m}(x', \Omega) \varphi_n(x', \Omega)}{n^2(x', \Omega)}$$

(24)

parametrically dependent on $\Omega$ and acting on the properly truncated $N$-dimensional vector space of complex amplitudes $a^\mu = (a_1^\mu, a_2^\mu, ..., a_N^\mu)$. This equation can in general not be satisfied for a real value of $D_0$ except at discrete values of $\Omega$. $T^{(0)}(\Omega)$ is a non-hermitian matrix and has $N$ complex eigenvalues $\lambda_n(\Omega)$ for general values of $\Omega$. As $\Omega$ is varied, the eigenvalues $\lambda_n(\Omega)$ flow in the complex plane (see Fig. 3), each one crossing the positive real axis at a specific $\Omega_n$, determined by $\text{Im}[\lambda_n(\Omega = \Omega_n)] = 0$. The modulus of the eigenvalue defines the “non-interacting” lasing threshold corresponding to that eigenvalue, $D_{th}^{(n)} = 1/\lambda_n(\Omega_n)$, the real frequency $\Omega_n$ is the non-interacting lasing frequency, and the eigenvector $a_n$ gives the “direction” of the lasing solution in the space of CF states. Among these solutions, the smallest $D_{th}^{(1)}$ (i.e., the largest of the real eigenvalues $\lambda_n$) gives the actual threshold for the first lasing mode; the frequency $\Omega_1$ is the lasing frequency at threshold and the eigenvector $a_1$ defines ”direction” of the lasing solution at threshold. The “length” of $a_1$ cannot be determined from the linear equation (23) but rises continuously from zero at threshold and is determined by the non-linear equation (22) infinitesimally above threshold. As noted, the remaining real eigenvalues of $T^{(0)}(\Omega)$ define the non-interacting thresholds for other modes, however the actual thresholds of all higher modes will differ substantially from their non-interacting values due to the non-linear term in (22) which now comes into play. The actual lasing frequencies of higher modes have a relatively weak dependence on $D_0$ and differ little from their non-interacting values. In Fig. 3 we show the flow of the eigenvalues of $T^{(0)}(\Omega)$ for the case of a 1D cavity, with the non-interacting thresholds and lasing frequencies indicated.

SALT has been successfully used to calculate the steady-state lasing characteristics of arbitrary 1D cavities, 2D uniform dielectric cavities of general shape, and 2D disordered media embedded in a disk-shaped gain medium.

3.2. Calculation of Purcell factors for open cavity structures

Spontaneous emission is a ubiquitous phenomenon that is responsible for most of the light around us. Any optically active emitter that is prepared in high energy states decays to lower energy states by emitting a photon through the action of various sources of noise. The most basic model of spontaneous emission considers the case of a two-level system (TLS) where the noise is caused by quantum fluctuations of electromagnetic vacuum that causes transitions between the two levels. In the perturbative regime, the spontaneous emission rate $\Gamma_{sp}$ can be calculated via the
Fermi Golden rule and related to the local density of states (LDOS) of photons at the emitter position $x_e$, $\rho(x_e, \omega_a)$

$$\Gamma_{sp} = \frac{\pi \omega_a P_{12}^2}{\hbar \epsilon_0} \rho(x_e, \omega_a)$$

Here $P_{12}$ is the electric dipole moment and $\omega_a$ is the transition frequency of the TLS. In free space (in three dimensions) $\rho = \omega_a^2 / 3\pi^2$ and we get the vacuum spontaneous emission rate

$$\Gamma_{sp}^{vac} = \frac{\omega_a^3 P_{12}^2}{3\pi \hbar \epsilon_0}$$

It was Purcell\(^7\) who pointed out that the spontaneous emission rate of an emitter can be dramatically modified in the presence of a cavity. The action of cavity can simply be thought of as modifying the LDOS of photons. Note that although spontaneous emission itself is due purely to the quantum nature of light, the spontaneous emission rate can be related to a purely classical quantity, the LDOS, which in turn can be related to the Green’s function of the wave equation that we calculated in Section 2:

$$\rho(x_e, \omega_a) = \frac{1}{\pi} \text{Im} [G(x_e, x_e; \omega_a)]$$

Once the CF modes of a cavity are determined, the spontaneous emission rate can be calculated readily using this expression. For illustration, we calculate in Fig. 4 the Purcell enhancement, $\Gamma_{sp} / \Gamma_{sp}^{vac}$ for a one dimensional dielectric cavity.

One striking feature in this plot is the strongly non-Lorentzian form of each of the peaks (to avoid confusion, we emphasize that what is calculated and plotted is not the spectrum). This does not derive from interference between various terms of the spectral expansion in Eq. (27); it’s present in each individual term contributing to
(27). More significantly, individual terms in the sum typically change sign as $\omega_a$ is varied (due to the complex valued CF functions in the numerator), however the sum total remains positive.

![Graph](image)

Fig. 4. Purcell enhancement factor for an emitter with a transition frequency at $\omega_a$. The enhancement factor, plotted as a solid line, is shown for $\omega_a$ varying within a range $(0, 40)$. The emitter is located at $x_e = a/2$. The inset shows a comparison of a small range to a fit by a series of Lorentzian functions $\sum m \kappa_m / ((\omega_a - \nu_m)^2 + \kappa_m^2)$ (dashed line) with central frequencies and decay rates given by those of the QB modes, Eq. (16). For this plot, $a = 1$ and $n_0 = 2$.

In conclusion, Purcell enhancement factors for arbitrary photonic structures, including plasmonic systems, can straightforwardly be calculated once the CF modes are determined.

4. Appendix

Here, we compare the CF Green’s function Eq. (17) to the exact Green’s function

$$\left[ \frac{\partial^2}{\partial x^2} - n^2(x) \Omega^2 \right] G(x, x'; \Omega) = \delta(x - x')$$

that can be obtained in analytical form for the 1D case. In addition to the conditions at $x = 0$, $x = a$ and $x = \infty$, we have the additional conditions $G(x' - 0^+,x'; \Omega) = G(x' + 0^+,x'; \Omega)$, $\partial G(x,x';\Omega)/\partial x|_{x=x'-0^+} - \partial G(x,x';\Omega)/\partial x|_{x=x'+0^+} = 1$. The resulting Green’s function is given by

$$G(x, x'; \Omega) = -\frac{1}{n_0 \Omega} \frac{\sin(n_0 \Omega x') e^{i \Omega(x-a)}}{\Delta(\Omega)}$$

outside the cavity ($x > a$). Here

$$\Delta(\Omega) = n_0 \cos(n_0 \Omega a) - i \sin(n_0 \Omega a)$$
The field response to an impulse-source is given by Eq. (15). To calculate this integral, we analyze the analytic structure of the integrand. There is no pole at \( \Omega = 0 \), because the singularity is cancelled by the trigonometric functions in the numerator. All the poles are the zeros of \( \Delta(\Omega) \) and are given by \( \tan(n_0 \Omega a) = -in_0 \); these are identical to Eq. (16). The final expression for the impulse-response is given by

\[
e_\delta(x, t) = \theta(t - x + a - n_0(a - x')) \frac{2\pi}{n_0 + 1} \sum_k (-1)^k \left( \frac{n_0 - 1}{n_0 + 1} \right)^k \left[ \theta \left( t - (x - a) - n_0(a + x') - 2kn_0a \right) - \theta \left( t - (x - a) - n_0(a - x') - 2kn_0a \right) \right] (31)
\]

This is exactly the CF result Eq. (17).

4. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part I (McGraw-Hill, New York, NY, USA, 1953).