Polarization properties and dispersion relations for spiral resonances of a dielectric rod

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Dielectric microcavities based on cylindrical and deformed cylindrical dielectric resonators have been employed as resonators for microlasers. Such systems support spiral resonances with finite momentum along the cylinder axis. For such modes the boundary conditions do not separate, and simple TM and TE polarization states do not exist. We formulate a theory for the dispersion relations and polarization properties of such resonances for an infinite dielectric rod of arbitrary cross section and then solve for these quantities for the case of a circular cross section (cylinder). Useful analytic formulas are obtained using the eikonal (Einstein–Brillouin–Keller) method, which are shown to be excellent approximations to the exact results from the wave equation. The major finding is that the polarization of the radiation emitted into the far field is linear up to a polarization critical angle (PCA) at which it changes to elliptical. The PCA always lies between the Brewster’s and total internal-reflection angles for the dielectric, as is shown by an analysis based on the Jones matrices of the spiraling rays. © 2005 Optical Society of America

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1. INTRODUCTION

There has been a great deal of recent interest in cylindrical and deformed cylindrical dielectric resonators for microlaser applications.1–5 From the theory side there is a particular interest in the deformed case, as in this case such resonators are wave-chaotic systems and can be analyzed with methods from nonlinear dynamics and semiclassical quantum theory. Analysis of the resonances and emission patterns from such systems has focused exclusively on the scalar Helmholtz equation that describes the axial component of the electric (TM mode) or magnetic (TE mode) field for the case of resonant modes with zero momentum in the axial direction (z direction). For this case (kz = 0) the polarization state is unchanged by boundary scattering, and the nontrivial ray dynamics in the transverse plane does not affect the polarization state of the resonant solutions. However, it is interesting to consider the solutions of the wave equation for both cylindrical and deformed cylindrical dielectric rods with kz ≠ 0 in that in this case the boundary scattering couples the electric and magnetic fields, and there no longer exist TM or TE solutions with a fixed direction in space for one of the fields. We refer to these nonzero kz modes as spiral modes and note that elastic scattering from such spiral resonance modes has been measured previously by Poon et al.6; the polarization properties were, unfortunately, not fully explored in those experiments.

The authors did, however, measure a systematic blue-shift of the resonance modes with tilt angle, which has been predicted by Refs. 7 and 8 and which we will derive below. The modes we study here are similar to the hybrid modes known in the study of optical fibers, in which it is also well known that there are no simple TE-, TM-, or TEM-like modes, but more complex vector solutions are necessary. Our emphasis differs, however, in several ways. First, we are interested in uniform dielectric rods, not the variable index profiles typical of optical fibers. Second, we are interested in modes that are not totally internally reflected so we can study the nature of the polarization of the emitted radiation in the far field. Third, we are primarily interested in resonances of uniform rods with cross sections in the range of tens to hundreds of micrometers, so they are strongly multimode and can be treated within the eikonal (semiclassical) approximation.

Here, we will model finite resonators as infinite dielectric rods, neglecting the end effects in the z direction; we therefore formulate the vector wave equation and the necessary boundary conditions for an infinite rod of arbitrary cross section. Various approximations are possible to treat end effects when they are relevant, but we will not explore them here. We define the resonant solutions (quasibound modes) of such a system and write down a general formalism that can be used to obtain exact numerical solutions for the vector resonances. We also show how the corresponding solutions can be used to derive the spatially varying polarization state of the emitted radiation in the far field. We then study in detail these equations in the case of a circular cross section (cylinder) for which a great deal of analytic progress and physical insight may be obtained. The analysis of the noncircular case for which wave-chaotic progress and physical insight may be obtained. The analysis of the noncircular case for which wave-chaotic polarization states are possible will be published elsewhere.9

Our goal is to relate the polarization state in the far field to the projected two-dimensional ray motion in the
plane transverse to the z axis (see Fig. 1). One can have spiral resonances that range from motion along the diameter of the rod in the transverse plane (bouncing-ball-type modes) to whispering-gallery modes that circulate around the perimeter of the rod as they spiral along it. We will discuss how the polarization properties of the resonances vary as the angle of incidence in the plane (\(\sin \chi\)) and the spiral (tilt) angle (\(\theta\)) with respect to the \(x-y\) plane varies. [In Fig. 2(a) we introduce the relevant geometric parameters.] It should be noted that, owing to the curvature of the boundary, even modes that are totally internally reflected according to geometric optics do emit by evanescent radiation into the far field, and their polarization fields can be obtained from exact solution of the wave equation, although experimentally it may be impractical to measure their weak emission far above the critical angle. In the simplest case of \(k_z = 0\) (\(\theta = 0\)), one finds pure linear polarization in the far field; in addition, the resonant energies are just those of the two-dimensional problem of a dielectric disk. In this two-dimensional case the resonant energies in the semiclassical limit are determined by the optical path length as well as by the phase shifts due to caustics and reflections at the boundary. These boundary terms in the semiclassical limit correspond simply to the total-internal-reflection (TIR) phase shifts for TM and TE scattering off a plane dielectric interface when \(\sin \chi > \sin \chi_c = 1/n\) (here \(n\) is the index of refraction of the rod surrounded by air); if \(\sin \chi < \sin \chi_c = 1/n\), there is zero phase shift but just a loss (imaginary part of \(k\)) given by the Fresnel scattering coefficients. For the spiral modes (\(k_z \neq 0\)), the boundary terms have a new character corresponding neither to the TE nor to the TM Fresnel scattering, and new phenomena can occur, such as a nonzero phase shift for modes that are not TIR. We derive below the generalization of the Fresnel scattering coefficients for spiral modes of the cylinder by using the vector eikonal method. We find that the angle at which a nonzero phase shift is set in is always between the critical angle and the Brewster's angle and coincides precisely with the onset of elliptical polarization in the far field. We call this new key quantity the polarization critical angle (PCA)

In Section 2 of the paper we set up the relevant form of the vector wave equations for the infinite dielectric rod and formulate the boundary conditions for the quasi-bound modes (resonances). In Section 3 we discuss how to extract the far-field polarization of these quasi-bound modes. All results are for the general case of arbitrary cross section of the rod. In Section 4 we specialize to the case of the dielectric cylinder and reduce the resonance problem to a simple root-finding problem. This equation is exact and is shown to yield the systematic blueshift in Ref. 6. In Section 5 we reformulate the same problem using the eikonal [Einstein–Brillouin–Keller (EBK)] method, which yields a simpler analytic formula for the
resonances and allows a statement of the polarization problem in terms of generalized Fresnel coefficients. The exact and EBK resonance wave vectors are shown to agree quite well, down to small $k$. In Section 6 we restate the polarization problem in terms of Jones matrices and thus derive the internal polarization state and the far-field polarization of the resonances in the semiclassical limit. Both the Jones and the EBK formulations are shown to yield the same answer for this quantity and to agree with the exact results to a good approximation. Finally, the origin of the PCA is explained.

2. WAVE EQUATION AND RESONANCES FOR THE INFINITE DIELECTRIC ROD

For electromagnetic fields in free space interacting with uniform dielectrics, Maxwell’s equations reduce to the vector Helmholtz equation

$$\nabla^2 \mathbf{E}(x,y,z) + k^2 \mathbf{E}(x,y,z) = 0,$$

(1)

where $n$ is the uniform index of the rod and $n_{\infty} = 1$ outside the rod. Thus $n$ differs from unity in an arbitrary closed simply-connected domain $\partial D$ in the $x-y$ plane for all values of $z$ (the rod is infinite in the $z$ direction). The translational symmetry along the $z$ axis (see Fig. 1) allows us to express the $z$ variation of the fields as

$$\mathbf{E}(x) = \mathbf{E}(x,y)\exp(-ik_z z), \quad \mathbf{B}(x) = \mathbf{B}(x,y)\exp(-ik_z z);$$

(2)

henceforth, the vectors $\mathbf{E}$, $\mathbf{B}$ will refer to the $x$, $y$-dependent vector fields just defined. With this ansatz, we can show that the most general solution of the six-component vector wave equation for this problem is determined by the $E_z$ and $B_z$ components alone; the perpendicular fields are given by linear combinations of these two scalar fields and their derivatives. Hence we must solve the two-component wave equation

$$\nabla^2 \mathbf{E}_z(x,y) + \gamma^2 \mathbf{E}_z(x,y) = 0,$$

(3)

with $\gamma^2 = n(x^2)k^2 - k_z^2$, where we have introduced the reduced wave vector $\gamma$, which is the wave vector associated with the transverse mode. The complication of solving this remaining two-component Helmholtz equation stems from the fact that the two fields $E_z$, $B_z$ are coupled through the boundary conditions. Four independent boundary conditions are found through the application of the general Maxwell boundary conditions:

$$E_{z1} = E_{z2}, \quad B_{z1} = B_{z2}, \quad \partial_{n} E_{z1} = \partial_{n} E_{z2}, \quad \partial_{n} B_{z1} = \partial_{n} B_{z2},$$

(4)

$$\frac{k}{\gamma_1} \partial_n B_{z1} - \frac{k}{\gamma_2} \partial_n B_{z2} = - \frac{k_z^2}{\gamma_1} \partial_n E_{z1},$$

(6)

Here, subscripts 1 and 2 refer to inside and outside solutions. We recover the familiar special case of two-dimensional modes when we take $k_z = 0$; in this case the boundary conditions can be satisfied with either $E_z = 0$ (TM solutions) or $E_z = 0$ (TE solutions); i.e., we get spatially uniform eigenpolarization directions. For the TM case, we find $E_z = 0$ and

$$B_z = \frac{i}{\gamma^2} n^2 k \left( \frac{\partial E_z}{\partial n} \right),$$

(8)

and, for the TE case, we find $B_z = 0$ and

$$E_z = - \frac{i}{\gamma^2} k \left( \frac{\partial B_z}{\partial n} \right).$$

(9)

In both cases Eqs. (4)–(7) completely decouple, and we have only to solve the scalar Helmholtz equation for $E_z$ or $B_z$ with the boundary conditions of the continuity of the field and its normal derivative on the boundary. In both cases the electromagnetic field is linearly polarized in the far field, either in the direction parallel to the rod axis (TM) or perpendicular to it (TE). We will now focus on the $k_z \neq 0$ modes, which we will refer to as spiral modes. We will not consider the extreme case where $k = k_z$ (TEM modes).

Our interest is in the dielectric rod as a resonator, i.e., as a device for trapping light. Experiments on resonators fall into two broad categories, and the presence of quasibound modes are manifested differently in these two situations. One can measure elastic scattering of incident laser light from such a rod, as was done in Ref. 6, and focus on the specific wave vectors at which one observes scattering resonances. For this case the linear wave equation that we are studying provides an exact description. One can also imagine the rod containing a gain medium and, when pumped, emitting laser light into these spiral resonances. In such a case, the linear wave equation is not an exact description; however, for high-$Q$ resonances, typically the resonances of the passive and active cavities are similar. For the laser to emit specifically into spiral modes, there would have to be some mechanism to suppress lasing of planar ($k_z = 0$) modes; one could imagine doing this with a small seed pump that is tuned to the frequency of a spiral mode and pushes it above threshold before all the other modes. It is possible to describe these two different physical situations (elastic scattering and lasing) by using appropriate boundary conditions on the linear vector Helmholtz equation. To be precise, we assume that the rod is bounded by the interface $\partial D$ given by (see Fig. 1)

$$\partial D = R(z, \phi) \quad \forall \ z \in \mathbb{R}, \ \phi \in [0, 2\pi].$$

(10)

We are typically interested in boundaries that are smooth and not too far from a circle; hence it is natural to expand the internal and external solutions $E_z$ and $B_z$ of Eq. (3) in cylindrical harmonics:
\[
\begin{align*}
E_{z}^{+} & = \sum_{m=-\infty}^{\infty} \left[ \frac{\alpha_{m}}{\xi_{m}} H_{\text{in}}^{m}(\gamma r) + \frac{\beta_{m}}{\eta_{m}} H_{\text{out}}^{m}(\gamma r) \right] \exp(im\phi), \\
B_{z}^{+} & = \sum_{m=-\infty}^{\infty} \left[ \frac{\nu_{m}}{\xi_{m}} H_{\text{in}}^{m}(\gamma r) + \frac{\delta_{m}}{\eta_{m}} H_{\text{out}}^{m}(\gamma r) \right] \exp(im\phi),
\end{align*}
\]

where \( H^{+} \) are the Hankel functions, \( H^{-} \) representing an incoming wave from infinity and \( H^{+} \) representing an outgoing wave.

To describe the scattering experiment, we simply apply the boundary conditions Eqs. (4)-(7), which will connect these interior and exterior solutions by a set of linear equations for the coefficients \( \alpha_{m}, \xi_{m}, \ldots, \eta_{m} \). These linear equations will have solutions for any incident wave vector \( k \) and will define a scattering matrix for the fields that can be used to calculate the intensity scattered at any given far-field angle. An example of such a calculation for the cylinder is given in Fig. 3; the values of \( kR \) (\( R \) is the cylinder radius) at which rapid variation is observed are the resonant wave vectors for which the incident light is trapped for long periods. However, the precise pattern of radiation in the far field in this case is determined by both the scattered and the incident radiation and is not representative of a lasing mode for which there is no incident radiation. To determine the resonances corresponding to emission from a source, it is conventional to use the Sommerfeld or radiation boundary conditions; in this case we would set the incident waves from outside to zero [the coefficients \( \delta_{m}, \eta_{m} \) in Eq. (12)] and still impose the boundary conditions of Eqs. (4)-(7). The resulting linear equations for the remaining coefficients would not have any solutions for real \( k \) (because current is not conserved) but would have discrete solutions for complex \( k \) values; those solutions are known as the quasi-bound modes of the problem. It can be shown that the discrete complex solutions of this problem correspond to the poles of the \( S \) matrix.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3}
\caption{Comparison of scattering and emission pictures for quasi-bound modes. The complex quasi-bound mode frequencies are plotted on the \( \text{Re}[kR]-\text{Im}[kR] \) plane. On the back panel we plot the real \( k \) \( S \)-matrix, scattering cross section at 170° with respect to the incoming wave direction. Notice that the most prominent peaks in scattering intensity are found at the values of \( k \) where a quasi-bound mode frequency is closest to the real axis. These are the long-lived resonances of the cavity. Also visible is the contribution of resonances with shorter lifetimes (higher values of \( \text{Im}[kR] \)) to broader peaks and the scattering background. Calculations are for a dielectric cylinder with \( n=1.5 \) and for \( k_{z} \).
\end{figure}

The far field is determined to give the emission pattern and polarization properties of the emitted radiation. As is well known, the Hankel functions with complex \( k \) and large argument grow exponentially and do not provide normalizable fields at infinity. This is unimportant for studying the emission patterns as a function of far-field angle or polarization properties, although it may cause some practical difficulties in numerical algorithms. If desired, this unphysical feature of the solutions can be avoided for a given resonance by one’s adding a tunable imaginary part of the index of refraction to yield a solution with real \( k \) outside the dielectric. This imaginary part represents linear amplification in the medium and would give an estimate for the lasing threshold for that mode if mode competition effects were negligible. One finds that these real \( k \) solutions are continuously related to the complex-\( k \) quasi-bound states at real index and have approximately the same spatial properties except for the absence of growth at infinity.

3. POLARIZATION OF SPIRAL MODES IN THE FAR FIELD

Having outlined how to solve for the quasi-bound spiral modes, we now analyze their polarization properties. Strictly speaking, polarization of the time-harmonic electromagnetic field cannot be defined inside the cavity or in the near field around it as the electric and magnetic fields need not be perpendicular to each other or to the direction of energy flow. In the far field, on the other hand, where the radiation is well approximated locally by a plane wave, we should be able to analyze the polarization of the emission from spiral modes in conventional terms. Assuming the matching problem is solved, the coefficients \( \nu_{m}, \xi_{m} \) determining \( E_{z}^{-}(\rho, \varphi), B_{z}^{-}(\rho, \varphi) \) are known, and these fields can be differentiated in order to find \( E_{\perp} \) everywhere outside the rod. These relations simplify if we use the large argument expansion of the Hankel functions and their recursion relations to find the relative magnitudes of the \( E_{\perp} \) field components in cylindrical coordinates as \( \rho \to \infty \),

\[
\begin{bmatrix}
E_{\varphi} \\
E_{\theta} \\
E_{z}
\end{bmatrix}
= \begin{bmatrix}
k_{z} \gamma_{z} & \sum_{m} \nu_{m} \exp[im(\varphi - \pi/2)] \\
- \frac{k_{z}}{\gamma_{z}} & \sum_{m} \xi_{m} \exp[im(\varphi - \pi/2)] \\
0 & \sum_{m} \nu_{m} \exp[im(\varphi - \pi/2)]
\end{bmatrix}
\begin{bmatrix}
k_{z} \gamma_{z} E_{\varphi} \\
- \frac{k_{z}}{\gamma_{z}} E_{\theta} \\
E_{z}
\end{bmatrix}
\]

\begin{equation}
(13)
\end{equation}
To extract the polarization at a particular angular direction \( \phi \), we need to recognize that far away from the rod the radiation is not propagating in the radial (\( r \)) direction with respect to the cylindrical coordinates centered on the rod axis but is instead propagating at angle \( \alpha \) between the \( r \) and the \( z \) directions determined by Snell’s law for the \( z \) motion (see Fig. 2). We can then rotate our coordinate system by \( \alpha \):

\[
\begin{bmatrix}
\cos \alpha & 0 - \sin \alpha \\
0 & 1 \\
\sin \alpha & 0 - \cos \alpha \\
\end{bmatrix}
\begin{bmatrix}
\tan aE_z \\
\sec aB_z \\
E_z \\
\end{bmatrix} = 
\begin{bmatrix}
0 \\
\sec aB_z \\
\sec aE_z \\
\end{bmatrix}.
\]

In this rotated coordinate system in the far field, the electric field (on the right-hand side) has only two components in the plane transverse to the propagation direction. Thus, the polarization of the electric field in the far field is then determined by the ratio of these two components, \( E_z(\phi)/B_z(\phi) \). If these two field amplitudes have zero phase difference, we have linear polarization in a certain direction that can vary as the angle of observation \( \varphi \) varies; if there is a nonzero phase shift \( \Delta \) between them, then we typically have elliptical polarization except in the special case of \( \Delta = \pi/2 \) and \( |B_z|^2 = |E_z|^2 \) corresponding to circular polarization. It should be noted that when the angle \( \alpha \) is complex \( k_z > k \), then the wave vector \( \gamma_z \) of the outgoing Hankel functions is pure imaginary and there is no propagating radiation as \( r \to \infty \); this corresponds to the TIR condition with respect to the \( z \) motion of an internal ray in the cylinder, \( n \sin \theta = 1 \) (see Fig. 2), for which there is no evanescent escape. It should be emphasized, however, that this does not correspond to the true total reflection condition for spiraling rays, which comes at smaller \( \theta \) except in the case of normal incidence in the transverse plane. In particular, one can ask about the polarization states in the far field of spiral whispering-gallery modes, which emit solely by evanescent escape.

The analysis up to this point has been exact for a dielectric rod of arbitrary cross section, and the various formulas can be used to solve for both the resonance wave vectors and the polarization properties of the spiral modes of such a system. We have developed and implemented a numerical algorithm to do this\(^1\); we intend to describe the algorithm and present results for deformed cylinders in a subsequent paper.\(^9\) At this point we specialize to the problem of spiral modes of a cylinder for which a number of analytic techniques are possible that will allow us to develop a useful physical picture.

4. QUASI-BOUND RESONANCES IN THE CYLINDER

A. General Derivation

Focusing now on the case of the cylinder (circular cross section), we have the immediate simplification that the Helmholtz equation and boundary conditions separate in cylindrical coordinates, and we can look for solutions corresponding to a single component of the angular momentum, \( m \), instead of the sums in Eq. (11). In the following we will use the following notation:

\[
J_m := J_m(\gamma R), \quad H_m := H_m^* (\gamma R),
\]

where \( \gamma_i = \sqrt{n^2_k^2 - k_z^2} \), \( i \in \{1, 2\} \), and \( R \) is taken on the boundary of the domain. With this convention we can write the ansatz for the cylinder:

\[
\begin{align*}
E_z^\gamma(r; m, j) &= \alpha_m J_m(\gamma_m^j r) \exp(i m \varphi) \quad r < R, \\
B_z^\gamma(r; m, j) &= \xi_m J_m(\gamma_m^j r) \exp(i m \varphi) \quad r < R, \\
E_z^\gamma(r; m, j) &= \nu_m H_m^*(\gamma_m^j r) \exp(i m \varphi) \quad r > R, \\
B_z^\gamma(r; m, j) &= \xi_m H_m^*(\gamma_m^j r) \exp(i m \varphi) \quad r > R.
\end{align*}
\]

Here, \( j \) is the radial mode index enumerating the solutions for a given \( m \). Using the boundary conditions for the continuity of the field, Eqs. (4) and (5), we get the relations

\[
\begin{align*}


\end{align*}
\]

Using this, we can rewrite Eqs. (6) and (7) in the following form:

\[
\begin{align*}


\end{align*}
\]

where the angles are given following the convention in Figs. 2(b) and 2(c), with \( \tan \theta = k_z/\gamma_1 \), \( \theta \) is the interior angle of the ray spiraling up with respect to the \((x, y)\) plane, and \( \alpha \) the corresponding exterior angle. For this system to have a nontrivial solution, the determinant needs to vanish, resulting in
\[(1 - n^2) m^2 \sin^2 \theta = \frac{1}{J_m H_m} [\cos^2 \alpha H_m \delta \rho_m - \cos^2 \theta J_m \delta \rho H_m] \times \frac{1}{J_m H_m} [\cos^2 \alpha H_m \delta \rho J_m - n^2 \cos^2 \theta J_m \delta \rho H_m] - n^2 \cos^2 \theta J_m \delta \rho H_m] = G^{\text{TM}} G^{\text{TE}}, \quad (22)\]

where we have defined
\[G^{\text{TE}} = \frac{1}{J_m H_m} [\cos^2 \alpha H_m \delta \rho J_m - n^2 \cos^2 \theta J_m \delta \rho H_m],\]
\[G^{\text{TM}} = \frac{1}{J_m H_m} [\cos^2 \alpha H_m \delta \rho J_m - \cos^2 \theta J_m \delta \rho H_m]. \quad (23)\]

This form is useful because the left-hand side is independent of \(k\) and vanishes as \(\theta \to 0\) for all \(m\), yielding
\[0 = G^{\text{TE}} G^{\text{TM}} = [H_m \delta \rho J_m - n^2 J_m \delta \rho H_m] \quad (24)\]
\[\times [H_m \delta \rho J_m - J_m - \delta \rho H_m]. \quad (25)\]

The vanishing of the left bracket describes the resonance condition for the usual two-dimensional TE modes, and the vanishing of the right bracket describes that for the TM modes, so one recovers the correct limiting behavior as \(\theta \to 0\) \((k_z \to 0)\). To find the resonance wave vectors for the spiral modes at \(\theta \neq 0\), one needs to find the complex roots of Eq. (22). An example of such solutions is given in Fig. 4, where the roots were found by the SLATEC routine dnse. The series of resonances for a given value of \(m\) will be labeled by a second integer \(j\), which indexes the quantized radial momentum in the transverse plane.

One can obtain further insight into the solutions by noting that for \(k_z = 0\) and \(m \neq 0\) the TE and TM resonance values differ so that at a typical TM resonance, for example, the factor \(G^{\text{TE}}\) will have its typical order of magnitude whereas the factor \(G^{\text{TM}}\) vanishes. By continuity of these functions with \(\theta\), we can expect for small \(\theta\) that the resonances will have one of the factors \(G^{\text{TM,TE}}\) small while the other is not and that the corresponding resonance will have a TM or TE character; i.e., the electric field will be predominantly in the \(z\) direction or predominantly in the \(x-y\) plane. The TE-like resonances will then show a large width (imaginary part) near the Brewster’s angle, and the TM-like resonances will not. As the tilt angle \(\theta\) increases, the spiral modes become full mixtures of TE and TM modes, and it is no longer possible to classify them in this manner. In Fig. 4, \(\theta\) is small enough to classify them as TE- and TM-like and the shading represents this classification.

Once the resonances are found, one can determine the polarization in the far field by rewriting Eq. (21) using the functions \(G^{\text{TM,TE}}\) as
\[\xi_m = \frac{m(n - n^3)\sin \theta}{n^2 G^{\text{TE}}} \alpha_m, \quad (26)\]
\[\alpha_m = \frac{m(n^3 - n)\sin \theta}{n^2 G^{\text{TM}}} \xi_m. \quad (27)\]

The coefficients \(\alpha_m, \xi_m\) determine the ratio of \(E_z\) to \(B_z\) in the far field by
\[P = \frac{B_z}{E_z} = \frac{\xi_m}{\alpha_m} = \frac{m(n - n^3)\sin \theta}{n^2 G^{\text{TE}}}. \quad (28)\]

We will, however, defer detailed analysis of the polarization properties of spiral resonances of the cylinder until we have developed the theory in a more intuitive semiclassical approximation in Section 5 below.

B. Small \(\theta\) Expansion

In the limit of small \(k_z\), or \(\theta\), we see that Eq. (22) is of order \(\theta^2\) on the left-hand side. Close to a TE- or TM-like resonance at \(k = k_0\), \(k_z = 0\), one of the factors \(G^{\text{TE}}\) or \(G^{\text{TM}}\) is small while the other is not. We expand the small term (which vanishes on resonance) to lowest order in \(\theta^2\) and insist that the resonance condition is satisfied at a slightly shifted value of the resonance wave vector, \(nk = n(k_0 + \Delta k_0)\). One can then show that
\[\Delta k_0/k_0 = \frac{1}{2} \alpha \theta^2, \quad (29)\]

where \(\alpha\) is given by a ratio of Bessel functions, which is plotted in Fig. 5. Note that the coefficient \(\alpha = 1\) for relatively small \(\sin \chi\); exactly this value follows from the EBK quantization formula near normal incidence discussed in Section 5. This result can be compared with experiments done by Poon et al., in which a tilted optical glass fiber.
indicate the wavefront. The wavefront-matched path is only intercepted by one's unwrapping the circular fiber. The dashed lines represent the refractive index. (b) The spiral quadratic blueshift can be interpreted by unwrapping the circular fiber. The dashed lines indicate the wavefront. The wavefront-matched path is only $2\pi a \cos \theta$, therefore the resonances are quadratic blueshifted with the tilt angle. The figures are adapted from Poon et al.\textsuperscript{6}

![Diagram](image)

**Fig. 6.** (a) Schematic of wave front-matching argument. The internal spiral wave of a tilted optical fiber with respect to the incoming wave. The phase-matching condition between the spiral mode and the external incident wave reduces the effective cavity length for the spiral wave by a distance $d/n$, where $n$ is the refractive index. Fixing the quantum numbers of the resonances, this argument implies a quadratic blueshift as a function of the tilt angle $\theta$.

In Fig. 7 we compare the blueshift obtained from the exact numerical solution of Eq. (32) with the small $\theta$ expansion, Eq. (29). The agreement is quite good. We will see at the end of Section 5 that the EBK quantization will give the same quadratic blueshift.

**5. EINSTEIN–BRILLOUIN–KELLER QUANTIZATION CONDITIONS**

**A. General Derivation**

As already noted, using the quantized solutions of resonance condition Eq. (22), we can easily calculate the polarization in the far field. However, for getting insight into the dependence of the polarization on the internal ray motion, a more appealing approach is to use the eikonal method and the Einstein–Brillouin–Keller (EBK)-type formulation of the resonance conditions. The eikonal method, in general, refers to finding approximate solutions of the wave equation with a specific ansatz that is expected to be good in the short-wavelength limit ($k \to \infty$). The EBK method describes how to apply that ansatz to boundary-value problems, typically assuming Dirichlet or Neumann boundary conditions.\textsuperscript{15} The approach has been used, for example, to find approximate quantization formulas for the circular and elliptical billiard systems.\textsuperscript{16} Recently it was generalized to treat the scalar Helmholtz equation for a dielectric billiard in two dimensions.\textsuperscript{10} However, it was also emphasized that the EBK method works only for the small subset of boundary shapes for which the ray motion within the boundary is integrable,\textsuperscript{13} a point that goes all the way back to Einstein’s original paper in 1917.\textsuperscript{17} The motion of a ray within an infinite cylinder is also integrable (the energy and $z$ components of linear momentum and angular momentum are conserved), and a generalization of the EBK method should work in this case also. The necessary generalization is to introduce the boundary conditions appropriate for the coupled $E_z$ and $B_z$ components of the field; this will lead to a generalization of the Fresnel coefficients for a plane interface.

We study the vector Helmholtz Eq. (3) for $E_z(x,y)$, $B_z(x,y)$ in the semiclassical limit $k$, $\gamma_{1,2} \to \infty$; in this limit we expect the solutions to have rapid phase variations and relatively slow amplitude variations. The generalized EBK ansatz for the quasi-bound solutions of the vector Helmholtz Eq. (3) can be written as

$$\begin{align*}
\begin{bmatrix} E_z \\ B_z \end{bmatrix} &= \Psi(r) = A_1 \exp[i\gamma S_1(r)] + A_2 \exp[i\gamma S_2(r)],
\end{align*}$$

(30)

where $A_{1,2}$ are two-component vectors and $S$ is the eikonal. Note that all the functions are defined on the two-dimensional $x$–$y$ plane. Following Refs. 10 and 16, we can write the general quantization condition as

$$\gamma \oint_{\Gamma_i} \mathbf{d}q \cdot \nabla S = 2\pi \gamma_i + \Phi_i \quad i = 1,2.$$  

(31)

Here the quantity $\nabla S$ is the gradient of the phase functions $S_1$, $S_2$ considered as the two sheets of a double-
valued vector field defined on the cross section, \( l \), are integers, and \( \Gamma \) refers to a topologically irreducible set of loops. To avoid confusion in this section, we temporarily drop the subscript \( \gamma \rightarrow \gamma \). Keller showed that in order for the EBK solution to be single valued it is necessary that these loop integrals of the phase be quantized and that any two topologically inequivalent and nontrivial loops can be chosen. \( \Phi \) is a total phase shift due to caustics and boundary scattering; for the scalar two-dimensional Helmholtz equation in a circle, these phase shifts are known for Dirichlet, Neumann, and dielectric boundary conditions. For the case of a dielectric circle, the phase shifts are complex in general, representing either refraction out of the circle or the phase shift at the boundary due to TIR. To get the appropriate ray dynamics for the spiral modes, it is easily shown that the eikonal \( \eta \) must be identical to that of the two-dimensional circular billiard (i.e., \( k_z = 0 \)); the new feature here is an additional eigenvalue condition on the amplitude two-vector, which will determine the eigenpolarization directions. This will lead to a modification of the phase shifts \( \Phi \) with respect to the two-dimensional case.

Before discussing the latter point, we briefly review the quantization relations, assuming the \( \Phi \) are known. Two conventional loops for implementing the quantization conditions are shown in Fig. 8. The first loop, \( \Gamma_1 \), goes just outside the inner turning point of a ray of fixed angular momentum; VS points in the direction of the ray so this integral just yields the length of the caustic for this ray. No caustic surfaces are crossed, and the path does not touch the boundary, so the phase \( \Phi_1 \) = 0. Thus the first loop gives a relation equivalent to angular-momentum conservation:

\[
\sin \chi = \frac{m}{\gamma R},
\]

where we have replaced the integer \( l \) by \( m \) to conform to our earlier notation.

The second loop \( \Gamma_2 \) gives the quantization condition for the reduced wave vector \( \gamma \) in terms of the path length \( L \) of the loop (the vector field \( \nabla S \) is everywhere parallel to this path):

\[
\gamma L = \left[ 2 \cos \chi - 2 \left( \frac{\pi}{2} - \chi \right) \sin \chi \right] R = 2 \pi j + \Phi_2,
\]

where we have replaced the integer \( l \) by \( j \). This is exactly the relation we would find for the two-dimensional billiard problem, except that the transverse wave vector \( \gamma = (\frac{\pi}{2} k^2 - k_z^2)^{1/2} \) has replaced the full wave vector \( nk \) and that the appropriate phase shift \( \Phi_2 \) needs to be determined and will differ from the two-dimensional case.

B. Semiclassical Boundary Conditions for a Dielectric Rod

To treat the spiral modes of a dielectric cylinder in this approach, we need to project the three-dimensional boundary conditions corresponding to Snell’s and Fresnel’s laws into two dimensions. All the relevant angles for this projection are defined in Fig. 2. Because we are in the ray optics limit, we can regard the scattering of the eikonal from the boundary as the scattering of a plane wave from the tangent plane. Because of our assumption of only outgoing waves, it is sufficient to assume only an incident, reflected, and transmitted wave (see Fig. 9).

1. Generalized Snell’s Law

The form of each of the fields is given by

\[
\Psi^m = \begin{pmatrix} E_{x,m} \\ B_{x,m} \end{pmatrix} \exp(i \gamma S_m),
\]

with \( m \in \{i, r, t\} \). The gradient of the eikonal \( \nabla S \) gives the direction of the ray and is of constant length \( |\nabla S| = n \). We find

\[
\frac{\partial_S S^i = i \gamma_1 \cos \chi, \quad \frac{\partial_S S^r = -i \gamma_1 \sin \chi, \quad \frac{\partial_S S^t = i \gamma_2 \cos \sigma},
\]

\[
\frac{\partial_S S^i = i \gamma_1 \sin \chi, \quad \frac{\partial_S S^r = i \gamma_1 \sin \chi, \quad \frac{\partial_S S^t = i \gamma_2 \sin \sigma}.
\]

(35)

The first set of boundary conditions, the continuity of the field across the boundary, Eqs. (4) and (5), becomes

\[
\begin{pmatrix} E_{z,i} \\ B_{z,i} \end{pmatrix} \exp(i \gamma_1 S^i) + \begin{pmatrix} E_{z,r} \\ B_{z,r} \end{pmatrix} \exp(i \gamma_1 S^r) = \begin{pmatrix} E_{z,t} \\ B_{z,t} \end{pmatrix} \exp(i \gamma_2 S^t).
\]

(36)

In that these equations need to hold everywhere on the boundary, the phases need to be equal, thus yielding

\[
\gamma_1 S^i = \gamma_1 S^t = \gamma_2 S^t.
\]

(37)

Using the fact that the tangent components are continuous, we obtain

\[
\gamma_1 \sin \chi = \gamma_2 \sin \sigma \Rightarrow \sin \chi = \frac{\gamma_2}{\gamma_1} \sin \sigma.
\]

(38)

This equation can be identified as the projection of Snell’s law into the transverse plane. Note that when \( k_z = 0 \) we recover the usual result:
We can write the projected Snell’s law in a completely geometric fashion, noting $\gamma_1/\gamma_2 = \tan \theta/\tan \alpha$, 

$$\sin \sigma = \frac{\sin \alpha \cos \theta}{\cos \alpha \sin \theta} \sin \chi = f(\theta) n \sin \chi, \quad (40)$$

with the function $f(\theta)$ given by 

$$f(\theta) = \frac{\cos \theta}{(1 - n^2 \sin^2 \theta)^{1/2}} \left( \frac{1 - \sin^2 \theta}{1 - n^2 \sin^2 \theta} \right)^{1/2} \geq 1. \quad (41)$$

Hence, as is clear geometrically, the projected Snell’s law leads to TIR of the projected motion before the critical angle $\sin \chi = 1/n$ is reached in the plane (this is simply because the actual angle of incidence is steeper than the projected angle owing to the z-motion); the function $f(\theta)$, which determines the effective critical angle, is plotted in Fig. 10.

2. Generalized Fresnel Matrices

The kinematics of the projected ray motion has been determined above simply from the continuity of the tangential components of the fields; the transport of ray flux across the boundary will now be determined from the normal derivative boundary conditions:

$$\mathbf{B}^l \left( \begin{array}{c} E_x^l \\ B_z^l \end{array} \right) \exp(i \gamma_1 S^l) + \mathbf{B}^r \left( \begin{array}{c} E_x^r \\ B_z^r \end{array} \right) \exp(i \gamma_2 S^r) = \mathbf{B}^l \left( \begin{array}{c} E_x^l \\ B_z^l \end{array} \right) \exp(i \gamma_2 S^l), \quad (42)$$

where the matrices $\mathbf{B}$ are derived from the boundary conditions Eqs. (6) and (7) and given by the matrices

$$\mathbf{B}^l = \left[ \begin{array}{cc} (n - n^3) \sin \theta \cdot \partial_n & \cos^2 \alpha \cdot \partial_n \\ n^2 \cos^2 \alpha \cdot \partial_n & (n^3 - n) \sin \theta \cdot \partial_n \end{array} \right], \quad (43)$$

$$\mathbf{B}^r = \left[ \begin{array}{cc} 0 & n^2 \cos^2 \theta \cdot \partial_n \\ n^2 \cos^2 \theta \cdot \partial_n & 0 \end{array} \right]. \quad (44)$$

Here $\partial_n$ and $\partial_t$ are the normal and tangential derivatives. We can now relate the incoming field to the outgoing and the reflected by using the boundary conditions Eqs. (36) and (42):

$$\Psi^r = R\Psi^i, \quad (45)$$

where $R$ and $T$ are the general Fresnel matrices given by

$$R = (\mathbf{B}^l - \mathbf{B}^r) \mathbf{B}^l (\mathbf{B}^l - \mathbf{B}^r)^{-1}, \quad (46)$$

and, similarly,

$$T = (\mathbf{B}^l - \mathbf{B}^r) \mathbf{B}^r (\mathbf{B}^l - \mathbf{B}^r)^{-1}. \quad (47)$$

In the limit $\theta \to 0 \Rightarrow \alpha \to 0$, these $2 \times 2$ matrices become diagonal and take the form

$$R = \left[ \begin{array}{cc} n \cos \chi - \cos \sigma & 0 \\ n \cos \chi + \cos \sigma & \cos \chi \cdot n \cos \sigma \end{array} \right] \Rightarrow R = \left[ \begin{array}{cc} r_s & 0 \\ 0 & -r_p \end{array} \right], \quad (49)$$

$$T = \left[ \begin{array}{cc} 2n \cos \chi & 0 \\ n \cos \chi + \cos \sigma & \cos \chi \cdot n \cos \sigma \end{array} \right] \Rightarrow T = \left[ \begin{array}{cc} t_s & 0 \\ 0 & t_p \end{array} \right]. \quad (50)$$

We recognize the diagonal elements as the Fresnel coefficients for TM (denoted by subscript $s$) and TE (subscript $p$) plane waves incident on a dielectric interface. The diagonal nature of the matrices implies that for $\theta=0$ these polarization states are preserved. When $\theta \neq 0$, the matrices have off-diagonal elements, implying the mixing of polarization states on reflection (strictly speaking, these matrices mix $E_x$ and $B_z$ on reflection, which will be shown to be equivalent to rotating the local polarization).

C. Quantization Condition

Recall that we are applying the EBK quantization condition (31) to the two-vector $(E_z(x, y), B_z(x, y))$; when we integrate around the path in Fig. 8(b), which touches the boundary, we must impose the correct dielectric boundary conditions on this two-vector. In general, this changes the ratio of $E_z$ to $B_z$ and will lead to a multivalued solution as we complete the closed loop (for uniform index rods, only boundary scattering leads to the rotation of the two-vector). Therefore, in order for us to have a single-valued solution, the ratio of $E_z$ to $B_z$ must be unchanged on reflection; i.e., the two-vector $\Psi^l$ must be an eigenvector $a$ of the reflection matrix:

$$Ra = \Lambda a. \quad (51)$$

We thus see that for the spiral modes of the cylinder there are two allowed mixtures of TM and TE polarizations for each resonance labeled by angular momentum $m$ and wave vectors $\gamma_1$, $k_z$; the eigenvalues of the $R$ matrix $\Lambda = \exp(i \eta)$ will give the extra phase shift $\Phi_2$ needed to complete the EBK quantization condition in Eq. (33) ($\Phi_2 = \eta + \pi/2$, where the term $\pi/2$ comes from the caustic phase shift). As already shown above, at $\theta=0$ the $R$ matrix is diagonal, conventional TE and TM states are eigenvectors, and the eigenvalues are just the Fresnel reflection coefficients, $r_s$, $r_p$. These have the familiar property of being
The boundary have been combined in where the contributions of the phase shift and the loss at values of for a refracted wave. Having determined the two possible well-known functions of \( n_{\text{R}} \) internally reflected TM and TE waves are different and no refracted wave above TIR. The phase shifts for totally above TIR and purely phase shift above TIR. Hence one has pure reflection and refraction modified for the spiral modes.

The behavior of the eigenvalues \( \lambda = \exp(i\eta) \) of the matrix \( R \) for \( k_2 \neq 0 \) is shown as a function of the angle \( \sin \chi \) in Fig. 11. Note that \( \eta \) is, in general, complex, and we are plotting here its magnitude. The magnitude of the eigenvalues are different up to a point between the Brewster’s angle (vertical dotted line) and the critical angle (vertical dashed line). At this point the eigenvalues become complex conjugates of each other. We will call this point, the polarization critical angle (PCA) and will explain how it is related to the far-field polarization below. Hence spiral modes acquire a phase shift on reflection before they reach the critical angle. Not until the critical angle do \( \lambda_{1,2} \) lie on the complex unit circle as they should for TIR. Thus we have a new phenomenon, a phase shift for a refracted wave. Having determined the two possible values of \( \phi \), we can write the quantization condition for the spiral resonances as a transcendental equation:

\[
2\gamma \left( 1 - \frac{m^2}{\gamma^2} \right)^{1/2} + 2m \arcsin \left( \frac{m}{\gamma} \right) = 2\pi j + \frac{\pi}{2} + m\pi + \zeta + i \ln|\eta|, \tag{52}
\]

where \( j \) and \( m \) are integers, \( \lambda = r \exp(i\zeta) \), and we have used the angular-momentum quantization condition Eq. (32). We can simplify this result further to get an explicit solution for rays near normal incidence in the plane, so that \( \sin \chi \to 0 \):

\[
\gamma = \pi \left[ j + \frac{m}{2} + \frac{1}{4} \right] + f(\chi, \theta), \tag{53}
\]

where the contributions of the phase shift and the loss at the boundary have been combined in \( f(\chi, \theta) \). We will analyze the function \( f(\chi, \theta) \) further in Section 6 on polarization properties of the spiral modes.

From Eq. (53) we can derive the blueshift for small \( \theta \) by noting that the right-hand side without \( f(\chi, \theta) \) is just the resonant condition for the circle. From the definition of \( \gamma = \sqrt{nk^2 - k_2^2} = nk(1 - \sin^2 \theta) \), we can write

\[
nk = nk_0 \left( 1 + \frac{1}{2} \theta^2 \right), \tag{54}
\]

where \( nk_0 \) is the resonance condition for the circle \( \theta = 0 \). Comparing it with Eq. (29) above, we see that this implies that the coefficient of the small \( \theta \) quadratic blueshift should be \( \alpha = 1 \) for small \( \sin \chi \) just as we found in Fig. 5 above. As noted there, this coefficient changes slightly when the PCA is reached; this change is captured by the contribution from \( f(\chi, \theta) \) we have just neglected. In Table 1 we compare the resonances found by th exact-wavelength-matching method [Eq. (22)] and by the EBK method, finding good agreement for \( \theta = 0.1, 0.2 \).

In general, we can always calculate the \( R \) matrix via Eq. (47), find the eigenpolarization directions, and subsequently act with \( T \) on the internal eigenpolarizations to determine the corresponding polarization in the far field. A physically more transparent method to do this involves the reformulation of the problem in terms of the actual polarization vector rather than the two-vector \((E_z, B_z)\). Below, we will introduce an equivalent matrix description in terms of the Jones algebra, which we will subsequently show to be exactly equivalent to the \((E_z, B_z)\) description.

### 6. JONES FORMULATION OF POLARIZATION PROPERTIES

We introduce the parallel and perpendicular components of the electric field, \( E_p \) and \( E_s \), in the local coordinate system defined by the plane of incidence. The Jones vector\(^{19}\) that describes the local polarization is

| Table 1. Spiral Resonances of the Cylinder with \( n = 2 \), \( \theta = 0.1, 0.2^a \) |
|-----------------|-----|---------------------|
| \( m \) | Mode | \( kR \) | \( j \) | \( kR \) |
| 18 | TE | 100.52083 - 0.27170i | 55 | 100.520468 - 0.271695i |
| 20 | TE | 100.42571 - 0.27098i | 54 | 100.425346 - 0.270979i |
| 44 | TE | 100.06672 - 0.25117i | 43 | 100.066168 - 0.255107i |
| 74 | TE | 100.30341 - 0.20475i | 31 | 100.301095 - 0.204629i |
| 98 | TE | 101.41861 - 0.09114i | 23 | 101.382400 - 0.075236i |
| 5 | TM | 99.62235 - 0.59423i | 29 | 99.61990 - 0.59421i |
| 17 | TE | 99.86057 - 0.55418i | 23 | 99.866999 - 0.55433i |
| 20 | TM | 99.29851 - 0.56992i | 22 | 99.294733 - 0.56984i |
| 34 | TE | 100.17837 - 0.84848i | 16 | 100.20152 - 0.85155i |
| 39 | TE | 99.73148 - 1.35232i | 14 | 99.74088 - 1.36053i |
| 57 | TM | 99.76778 - 0.00005i | 8 | 96.79993 + 0.00009i |
| 62 | TM | 100.78974 - 0.00005i | 7 | 100.86339 + 0.00009i |

\(^a\)We compare the resonances calculated by the solution of Eq. (22) and the EBK method, finding good agreement. Although resonances can no longer be classified as TM or TE, classification as TM-like or TE-like is based on which factor \( G^{TE,TM} \) is small at the resonance as discussed above.
The angle $\phi$ on the cylinder must be the same for any value of $z$ and must be described by the same Jones vector when referred to the plane of incidence at that point. Therefore the Jones vector emerging from a reflection must be the same as the incident Jones vector once the coordinate system has been rotated into the new plane of incidence. This yields an eigenvalue condition for the Jones vectors describing the spiral modes of the cylinder:

$$\mathcal{J} \begin{pmatrix} E_p \\ E_s \end{pmatrix}_{1,2} = v_{1,2} \begin{pmatrix} E_p \\ E_s \end{pmatrix}_{1,2},$$

where $\mathcal{J} = \Re(\xi)J_r$. Thus the two polarization states are the two eigenvectors of $\mathcal{J}$, and their eigenvectors $v_{1,2}$ describe the phase shifts and refractive losses at each reflection, just as do the eigenvalues of $R$ for the two-vector $E_z, B_z$ studied earlier. Therefore, the two matrices $\mathcal{J}$ and $R$ are related by a similarity transformation and have the same eigenvalues,\textsuperscript{11} which is confirmed in Fig. 12.

### 7. POLARIZATION CRITICAL ANGLE

Realizing that the eigenvalues of interest are obtained from the product of a rotation and a diagonal Jones matrix with known entries allows us to understand the behavior of these eigenvalues rather simply. The eigenvalues can now be written in terms of the rotation angle of the plane of incidence, $\xi$, and the Fresnel reflection coefficients $r_s, r_p$:

$$v_{1,2} = \begin{pmatrix} 1,2 \end{pmatrix} \begin{pmatrix} r_s - r_p \cos \xi \pm \frac{1}{2} \sqrt{\cos^2 \xi (r_s - r_p)^2 + 4 r_s r_p} \\ 1,2 \end{pmatrix}$$

Consider the discriminant of the eigenvalues, $D = \cos^2 \xi (r_s - r_p)^2 + 4 r_s r_p$, which determines whether they are real or complex. Recall that the TM Fresnel reflection coefficient, $r_s$, is real and positive for all angles below the critical angle passing through unity and becoming complex and unimodular above the critical angle, whereas the TE reflection coefficient, $r_p$, is real below the critical angle but becomes negative at the Brewster’s angle and passes through negative 1 before becoming unimodular and complex above the critical angle. It follows that $D$ will always be positive and $v_{1,2}$ will be real for angles of incidence below the Brewster’s angle. However for any nonzero value of the rotation angle $\xi$, $D$ will become zero before the criti-

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**Fig. 12.** Comparison of the eigenvalues of the rotated Jones matrix, absolute value (solid gray and black curves) and the phase (divided by $\pi$)—(dashed gray and black curves) with the eigenvalues of $R$ (gray and black circles). Parameters of the calculation are $\tan \theta = 0.2$ and $n = 2$.

**Fig. 13.** (a) Absolute value of the two eigenvalues $v_{1,2}$ of the rotated Jones matrix (solid gray and black curves). The phase of the eigenvalues (divided by $\pi$) is plotted in dashes. The eigenvalues become complex at the point where the two curves meet and join. This point lies between the Brewster’s angle (dashed vertical black line) and the effective critical angle (dotted vertical black line). Calculated for $\tan \theta = 0.2$, $n = 2$. (b) Black curve, the sine of the PCA at which the eigenvalue of $R$ gets complex. Dotted curve, sine of the Brewster’s angle. Dashed curve, sine of the critical angle of TIR.
cal angle in that at the critical angle \( D = 4(1 - \cos^2 \xi) \) is negative. The value of the incidence angle \( \xi \) when \( D = 0 \) is the PCA, which we have already mentioned above. Because it occurs when both \( r_s \), \( r_p \) have absolute value less than unity, the eigenvalues \( \nu_{1,2} \) also have modulus less than unity, and we have a phase shift on reflection while a fraction \( 1 - |\nu_{1,2}|^2 \) of the incidence wave is refracted out. This fraction can be calculated and does not correspond to either of the usual Fresnel transmission coefficients for TM or TE. The behavior of the eigenvalues just described is shown in Fig. 13(a); the behavior of the PCA versus \( \sin \chi \) is shown in Fig. 13(b). We see that for small \( \theta \) the onset of the phase shift is close to the critical angle; as \( \theta \) varies, the PCA moves close to the Brewster’s angle, and then for \( \theta \) close to the critical angle, where for any \( \sin \chi \) we will have TIR, the PCA returns to the critical angle. This analysis allows us a simple understanding of why there is a PCA that precedes TIR. The TM and TE components of the spiral resonances have no relative phase shift at reflection until the Brewster’s angle; at the Brewster’s angle, the TE component picks up a \( \pi \) phase shift, which mixes with the TM component to give a phase shift between zero and \( \pi \). Right at the Brewster’s angle, \( r_p \) vanishes, and the local TE component of the resonance is filtered out, but as the TE reflectivity picks up above the Brewster’s angle this phase shift appears before the TIR condition is reached. It is interesting to note that, owing to the finite phase acquired at the PCA, the resonance condition Eq. (53) is slightly changed. The term PCA suggests that at this angle the polarization properties of the spiral modes change. We shall see that this is the case in Section 8.

### 8. Polarization Properties of Spiral Modes

When the eigenvalues of the \( J \) matrix are real, the eigenvectors can be chosen real (up to an overall phase), and thus there is no relative phase shift between \( E_s \) and \( E_p \). Although, in general, polarization is not well defined inside the dielectric, in the short-wavelength limit we are now examining it is well defined with respect to the considered ray direction, and zero relative phase shift implies linear polarization of the resonance with respect to the spiraling ray direction. Above the PCA, when the eigenvalues are complex, the eigenvectors also become complex, and there is a relative phase shift between \( E_s \) and \( E_p \) corresponding to elliptical polarization of the internal field. This picture is confirmed by the calculation of the eigenvalues plotted in Figs. 14(a) and 14(b). At small \( \sin \chi \) we have a large ratio between the eigenvector components and zero phase shift, corresponding to linear polarization with the electric field almost completely in the \( z \) or transverse direction (TM-like and TE-like). After the PCA we get a phase shift approaching \( \pi/2 \) and equal ratios leading to circular polarization right at the critical angle; above the critical angle we have, in general, elliptical polarization for the internal fields with a calculable ellipticity and with a phase difference of \( \pi/2 \) as expected for whispering-gallery modes.
One can find the far-field polarization arising from this internal field by applying to the relevant eigenvector of $\mathbf{J}$ the transmission matrix $\mathbf{J}_\theta$ and then rotating the resulting vector by an angle $\Theta$, which projects the Jones vector onto the plane perpendicular to the outgoing ray direction discussed in Section 6 above. This angle $\Theta$ can be determined by straightforward geometric considerations that we omit here and simply state that the far-field polarization vector $(E_x', E_y')$ is given by

$$\mathcal{R}(\Theta)\mathbf{J}_\theta|\alpha\rangle,$$

where $\mathcal{R}$ is the rotation just mentioned and $|\alpha\rangle$ is either of the eigenpolarization vectors of $\mathbf{J}$.

In Figs. 15(a) and 15(b), we compare the far-field polarization states for spiral resonances of the cylinder as predicted by the exact and geometric optics approach. We find good agreement between the methods, although the exact solutions smooth the abrupt behavior near the PCA predicted by the geometric optics approach. Above the critical angle, we have only evanescent emission, but the eccentricity of the two-vector is still finite. Note that as formulated, the Jones approach gives a continuous solution for the two polarization states, whereas the exact resonances are discrete and correspond to discrete allowed angles $\chi$, which can be also found through the EBK approach. The Jones approach provides a smooth and $k$-independent formula for the polarization states that agrees with those of the resonances.

9. SUMMARY AND CONCLUSIONS

We have reduced the Maxwell's equations for a dielectric rod of arbitrary cross section to a vector Helmholtz equation for a two-component vector field in the two-dimensional cross-sectional plane. We have devised a formulation of the resonance problem for the quasi-bound modes (spiral resonances), which can be implemented numerically for a general cross section, and shown how the polarization state of the resonances in the far field can be determined. Calculations were reported for the case of a circular cross section (cylinder), and the results were compared with ray-optical results from an EBK formulation of the resonance problem in the semiclassical limit. We have analyzed the polarization state of the spiral resonances both inside the cylinder and in the far field and related its properties to the internal ray motion. It was shown that, as the tilt angle of the spiraling ray with respect to the cross-sectional plane is increased, there exists a polarization critical angle at which the polarization changes from linear to elliptical both internally and externally, and this occurs before the total-internal-reflection condition, so the effect can be measured readily in the far field. The physical picture we developed in terms of the Jones polarization vectors was useful in understanding the PCA and may be useful in generalizing the analysis to arbitrary cross sections for which the ray motion can be chaotic.

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